DECORATED MARKED SURFACES: CALABI-YAU CATEGORIES AND RELATED TOPICS

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Abstract. This is a survey on the project 'Decorated Marked Surfaces' ([Q1, Q2, QZ2, BQZ, QZ3, Q3, KQ]), where we introduce the decoration Δ on a marked surfaces $\mathbf{S}$, to study Calabi-Yau-2 (cluster) categories, Calabi-Yau-3 (Fukaya) categories, braid groups for quivers with potential, quadratic differentials and stability conditions.

1. Introduction

We introduce the decorated marked surface from Fomin-Shapiro-Thurston's marked surface, which was originally studied in the cluster theory. The motivation comes from studying Calabi-Yau categories associated to quivers with potential and the corresponding spaces of Bridgeland stability conditions. This project is trying to understand categories via (surface) topology, which also fits into the framework [DHKK] that relates dynamical systems (i.e. quadratic differentials) and categories.

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2. Categories and topology

2.1. **Cluster categories and marked surfaces.** Following [FST], a marked surface $\mathbf{S}$ is a connected oriented smooth surface with a finite set $\mathbf{M}$ of marked points on its (non-empty) boundary $\partial\mathbf{S} = \bigcup_{i=1}^{b} \partial_i$ such that $m_i = \mid \partial_i \cap \mathbf{M} \mid \geq 1$. Such a marked surface is determined, up to diffeomorphism, by the numerical data g, b and the partition $m = \mid \mathbf{M} \mid = \sum_{i=1}^{b} m_i$. We have the following terminology:

- an *open arc* is (the isotopy class of) a curve on $\mathbf{S}$ with endpoints in $\mathbf{M}$ but otherwise in $\mathbf{S}^\circ = \mathbf{S} \setminus \partial\mathbf{S}$. Note that all curves are required to be simple and essential.
- two arcs are *compatible* if they do not intersect (except maybe at their endpoints),
- an *ideal triangulation* T of $\mathbf{S}$ is a maximal collection of compatible open arcs.

An elementary result (cf. [FST, Prop. 2.10]) is that any ideal triangulation T of $\mathbf{S}$, consists of $n = 6g - 6 + 3b + m$ open arcs and divides $\mathbf{S}$ into $\aleph = (2n + m)/3$ triangles. The *unoriented exchange graph* $\underline{\mathrm{EG}}(\mathbf{S})$ has vertices corresponding to ideal triangulations of $\mathbf{S}$ and edges corresponding to *flips*, as illustrated in the lower row of Figure 1.

Let $\mathbf{S}$ be a marked surface and T a triangulation of $\mathbf{S}$. Then there is an associated quiver Q_T with a potential W_T, constructed as follows (See, e.g. [QZ1] for the precise definition):

The survey is in a final form and no version of it will be submitted for publication elsewhere.

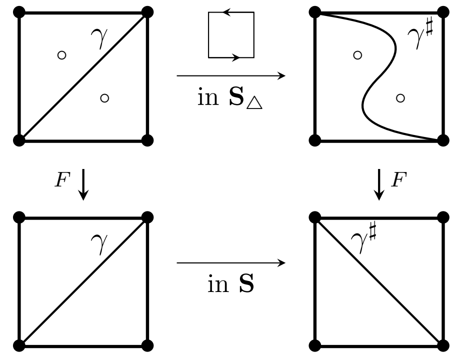

FIGURE 1. The decorated/ordinary forward flips

- the vertices of Q_T are (indexed by) the arcs in T;
- the arrows of Q_T are (anti-clockwise) angles of triangles in T;
- these three arrows form a 3-cycle in Q_T and W_T is the sum of all such 3-cycles.

The Ginzburg dg algebra $\Gamma_T = \Gamma(Q_T, W_T)$ is the Calabi-Yau-3 algebra constructed from the combinatorial data. There are three categories associated to Γ_T

- the *finite dimensional derived category* $\mathcal{D}_{fd}(\Gamma_T)$, which is Calabi-Yau-3;
- the *perfect derived category* $\operatorname{per}\Gamma_T$ (that contains $\mathcal{D}_{fd}(\Gamma_T)$);
- the *cluster category* $\mathcal{C}(\Gamma_T) := \operatorname{per}\Gamma_T/\mathcal{D}_{fd}(\Gamma_T)$, which is Calabi-Yau-2.

The marked surface $\mathbf{S}$ provides a good topological model for $\mathcal{C}(\Gamma_T)$. In fact, such a category is independent of the choice of T in the proper sense and thus will be denoted by $\mathcal{C}(\mathbf{S})$.

Theorem 1. [FST, BZ, QZ1] *There are following correspondences:*

(O) The rigid string objects in $\mathcal{C}(\mathbf{S})$ are characterized by open arcs on $\mathbf{S}$.
(M) The dimension of Ext^1 between rigid string objects in $\mathcal{C}(\mathbf{S})$ is given by the intersection numbers between the corresponding open arcs.
(EG) The exchange graph $\underline{\operatorname{EG}}(\mathbf{S})$ is the cluster exchange graph $\operatorname{CEG}(\mathbf{S})$ for $\mathcal{C}(\mathbf{S})$, whose the vertices/edges are cluster tilting objects/mutation (cf. [QZ1]).

2.2. **Calabi-Yau-3 categories and decorated marked surfaces.** The first aim of this project is to construct the good topological model for $\mathcal{D}_{fd}(\Gamma_T)$ and prove the analogue result as Theorem 1. Note that in [BQZ], we show that $\mathcal{D}_{fd}(\Gamma_T)$ is independent of T in a proper sense, thus we will denote it by $\mathcal{D}_{fd}(\mathbf{S}_\triangle)$.

Definition 2. The *decorated marked surface* $\mathbf{S}_\triangle$ is a marked surface $\mathbf{S}$ together with a fixed set $\triangle$ of $\aleph$ 'decorating' points in $\mathbf{S}^\circ$.

We also have open arcs in $\mathbf{S}_\triangle$ and a (decorated) triangulation $\mathbb{T}$ of $\mathbf{S}_\triangle$ is a collection of compatible open arcs that divides $\mathbf{S}_\triangle$ into $\aleph$ many once-decorated triangles. The corresponding (oriented) forward flip is shown in the upper row of Figure 1, where one moves the endpoints of an open arc anticlockwise along the quadrilateral to obtain a new open arc (to form the flipped triangulation). Denote by $\operatorname{EG}(\mathbf{S}_\triangle)$ the exchange graph of triangulations of $\mathbf{S}_\triangle$, whose vertices/edges are triangulation of $\mathbf{S}_\triangle$/forward flips. Note that there is an obvious forgetful map $F\colon \mathbf{S}_\triangle \to \mathbf{S}$, which is shown (vertically) in Figure 1. Then we have the following.

Theorem 3. [Q2] *There are following correspondences:*

- *(O) The reachable rigid string objects in* $\operatorname{per}\Gamma_T$ *are characterized by open arcs on* $\mathbf{S}_\triangle$.
- *(EG) Any connected component of the exchange graph* $\mathrm{EG}(\mathbf{S}_\triangle)$ *can be identified the principal connected component of the silting exchange graph for* $\operatorname{per}\Gamma_T$*, where the vertices/edges are silting objects/(forward) mutation (cf.* [Q2]*).*

Moreover, we have a new type of arcs:

- a *closed arc* is (the isotopy class of) a curve on $\mathbf{S}_\triangle$ with different endpoints in $\triangle$. See dashed arcs in the left picture of Figure 3.

This type of arcs plays a key role in braid groups and mapping class groups of surfaces. Namely, we have the following notions. The mapping class group $\mathrm{MCG}(\mathbf{S}_\triangle)$ is the group of isotopy classes of diffeomorphisms of $\mathbf{S}_\triangle$, where all diffeomorphisms and isotopies fix $\mathbf{M}$ and $\triangle$ setwise. On the other hand, the mapping class group $\mathrm{MCG}(\mathbf{S})$ fixes just $\mathbf{M}$ setwise. Thus there is a forgetful group homomorphism

$$F_M\colon\ \mathrm{MCG}(\mathbf{S}_\triangle) \to \mathrm{MCG}(\mathbf{S}), \tag{2.1}$$

whose kernel is the so-called surface braid group $\mathrm{SBr}(\mathbf{S}_\triangle)$. We are particularly interested in the braid twist group $\mathrm{BT}(\mathbf{S}_\triangle)$, which is subgroup of $\mathrm{SBr}(\mathbf{S}_\triangle)$, generated by braid twist along closed arcs, cf. Figure 2.

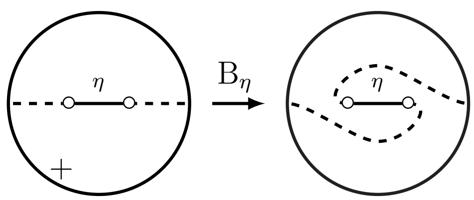

FIGURE 2. The Braid twist

Then we have the following.

Theorem 4. [Q1, QZ2] *There are following correspondences:*

- *(O) The reachable spherical objects in* $\mathcal{D}_{fd}(\mathbf{S}_\triangle)$ *are characterized by closed arcs on* $\mathcal{D}_{fd}(\mathbf{S}_\triangle)$.
- *(M) The dimension of* $\mathrm{Hom}^\bullet$ *between spherical objects in* $\mathcal{D}_{fd}(\mathbf{S}_\triangle)$ *is given by the intersection numbers between the corresponding closed arcs.*
- *(G) The braid twist group* $\mathrm{BT}(\mathbf{S}_\triangle)$ *can be identified with the (so-called spherical twist) subgroup* $\mathrm{ST}(\mathbf{S}_\triangle)$ *of the auto-equivalence group* $\operatorname{Aut}\mathcal{D}_{fd}(\mathbf{S}_\triangle)$*, where the braid twist along a closed arc becomes the spherical twist of the corresponding spherical object.*
- *(EG) Any connected component* $\mathrm{EG}^\circ(\mathbf{S}_\triangle)$ *of the exchange graph* $\mathrm{EG}(\mathbf{S}_\triangle)$ *can be identified the principal connected component* $\mathrm{EG}^\circ\mathcal{D}_{fd}(\mathbf{S}_\triangle)$ *of the exchange graph of* $\mathcal{D}_{fd}(\mathbf{S}_\triangle)$*, where the vertices/edges are hearts/simple (forward) tilting (cf.* [Q1]*).*

Remark 5. The isomorphism $\mathrm{BT}(\mathbf{S}_\triangle) \cong \mathrm{ST}(\mathbf{S}_\triangle)$ mentioned above plays a key role in understanding the generalization of braid group to the case of quivers with (super)potential, see [Q3] for details. Their (finite) presentations in [QZ3] also plays a crucial technical role the dissuasion below.

3. Quadratic differentials as stability conditions

One of the motivations to introduce decorated marked surface is coming from the interplay between quadratic differentials and categories.

3.1. **Framed uadratic differentials.** Let $\mathbf{X}$ be a compact Riemann surface and $\omega_{\mathbf{X}}$ be its holomorphic cotangent bundle. A *meromorphic quadratic differential* ϕ on $\mathbf{X}$ is a meromorphic section of the line bundle $\omega_{\mathbf{X}}^{\otimes 2}$. We consider *GMN differentials* ϕ on $\mathbf{X}$, which are meromorphic quadratic differential such that all zeroes of ϕ are simple and all poles of ϕ have order at least three. The real (oriented) blow-up of $(\mathbf{X}, \phi)$ is a differentiable surface $\mathbf{X}^{\phi}$, which is obtained from the underlying differentiable surface by replacing each pole P of ϕ by a boundary ∂_P, where the points on the boundary correspond to the real tangent directions at P. Furthermore, we will mark the points on ∂_P that correspond to the distinguished tangent directions, so there are $\mathrm{ord}_{\phi}(P) + 2$ marked points on ∂_P.

Definition 6. The decorated real blow-up $\mathbf{X}^{\phi}_{\triangle}$ of $(\mathbf{X}, \phi)$ is the decorated marked surface obtained from $\mathbf{X}^{\phi}$ by adding the set of zeroes of ϕ as decorations. Given any decorated marked surface $\mathbf{S}_{\triangle}$, an $\mathbf{S}_{\triangle}$*-framed quadratic differential* $(\mathbf{X}, \phi, \psi)$ is a Riemann surface $\mathbf{X}$ with a GMN differential ϕ, equipped with a diffeomorphism $\psi\colon \mathbf{S}_{\triangle} \to \mathbf{X}^{\phi}_{\triangle}$, preserving the marked points and decorations (see right picture of Figure 3)..

Two $\mathbf{S}_{\triangle}$-framed quadratic differentials $(\mathbf{X}_1, \phi_1, \psi_1)$ and $(\mathbf{X}_2, \phi_2, \psi_2)$ are equivalent, if there exists a biholomorphism $f\colon \mathbf{X}_1 \to \mathbf{X}_2$ such that $f^*(\phi_2) = \phi_1$ and furthermore $\psi_2^{-1} \circ f_* \circ \psi_1 \in \mathrm{Diff}_0(\mathbf{S}_{\triangle})$, where $f_*\colon \mathbf{X}_1^{\phi_1} \to \mathbf{X}_2^{\phi_2}$ is the induced diffeomorphism. Here $\mathrm{Diff}_0(\mathbf{S}_{\triangle})$ is the identity component of the group $\mathrm{Diff}(\mathbf{S}_{\triangle})$ of diffeomorphisms preserving marked points and decorations (each setwise).

Denote by $\mathrm{FQuad}(\mathbf{S}_{\triangle})$ the moduli space of $\mathbf{S}_{\triangle}$-framed quadratic differentials and take any connected component $\mathrm{FQuad}^{\circ}(\mathbf{S}_{\triangle})$ (which are isomorphic to each other).

Remark 7. The foliation of a quadratic differential gives (the so-called WKB) a triangulation of its decorated real below-up (see left picture of Figure 3). Therefore the exchange graph $\mathrm{EG}^{\circ}(\mathbf{S}_{\triangle})$ of triangulations is the skeleton of $\mathrm{FQuad}^{\circ}(\mathbf{S}_{\triangle})$, as $\mathbb{Z}^n$ is the skeleton of $\mathbb{C}^n$ (cf. Figure 4).

3.2. **Stability conditions.**

Definition 8 (Bridgeland)**.** A *stability condition* $\sigma = (Z, \mathcal{P})$ on a triangulated category $\mathcal{D}$ consists of a central charge $Z \in \mathrm{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C})$ and a collection of full additive subcategories $\{\mathcal{P}(\varphi) \subset \mathcal{D} \mid \varphi \in \mathbb{R}\}$, such that

- if $0 \neq E \in \mathcal{P}(\varphi)$ then $Z(E) = m(E) \exp(\varphi \pi \mathbf{i})$ for some $m(E) \in \mathbb{R}_{>0}$;
- $\mathcal{P}(\varphi + 1) = \mathcal{P}(\varphi)[1]$, for all $\varphi \in \mathbb{R}$;
- if $\varphi_1 > \varphi_2$, then $\mathrm{Hom}(\mathcal{P}(\varphi_1), \mathcal{P}(\varphi_2)) = 0$;
- any object $E \in \mathcal{D}$ admits a Harder-Narashimhan filtration of triangles (cf. [Q1, (8.1)]), whose factors are $A_j \in \mathcal{P}(\varphi_j)$ for $\leq j \leq m$ satisfying $\varphi_1 > \varphi_2 > ... > \varphi_m$.

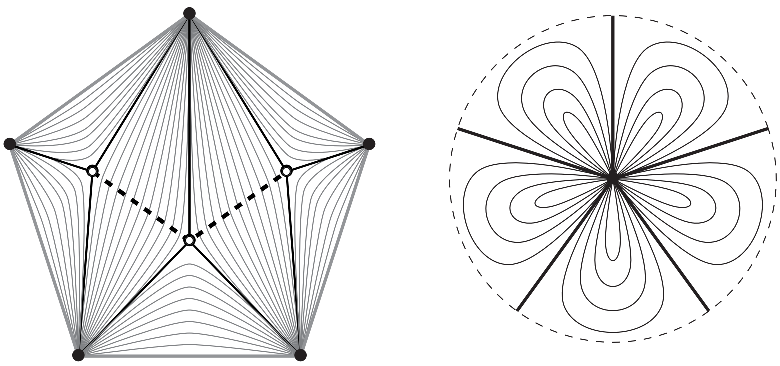

FIGURE 3. The foliation of a GMN differential on $\mathbb{P}^1$ for the A_2 case

Bridgeland shows that all stability conditions on $\mathcal{D}$ form a complex manifold $\operatorname{Stab}\mathcal{D}$ of dimension n with local coordinate Z for $n = \operatorname{rank} K(\mathcal{D})(< \infty)$. For $\mathcal{D}_{fd}(\mathbf{S}_\triangle)$, denote by $\operatorname{Stab}^\circ \mathcal{D}_{fd}(\mathbf{S}_\triangle)$ its principal connected component. As in Remark 7, exchange graph $\operatorname{EG}^\circ \mathcal{D}_{fd}(\mathbf{S}_\triangle)$ of hearts is the skeleton of $\operatorname{Stab}^\circ \mathcal{D}_{fd}(\mathbf{S}_\triangle)$. Such a philosophy/approach has been explored carefully (cf. the survey [Q4]). Building on Bridgeland-Smith's seminar work [BS], we prove the following.

Theorem 9. [KQ] *There is an isomorphism* $\operatorname{Stab}^\circ \mathcal{D}_{fd}(\mathbf{S}_\triangle) \cong \operatorname{FQuad}(\mathbf{S}_\triangle)$ *between complex manifolds and they are simply connected.*

Note that the correspondence (O) in Theorem 4 plays a key role in the isomorphism above: the saddle trajectories η of a quadratic differential, which are closed arcs, correspond to semistable(/spherical) objects for stability conditions.

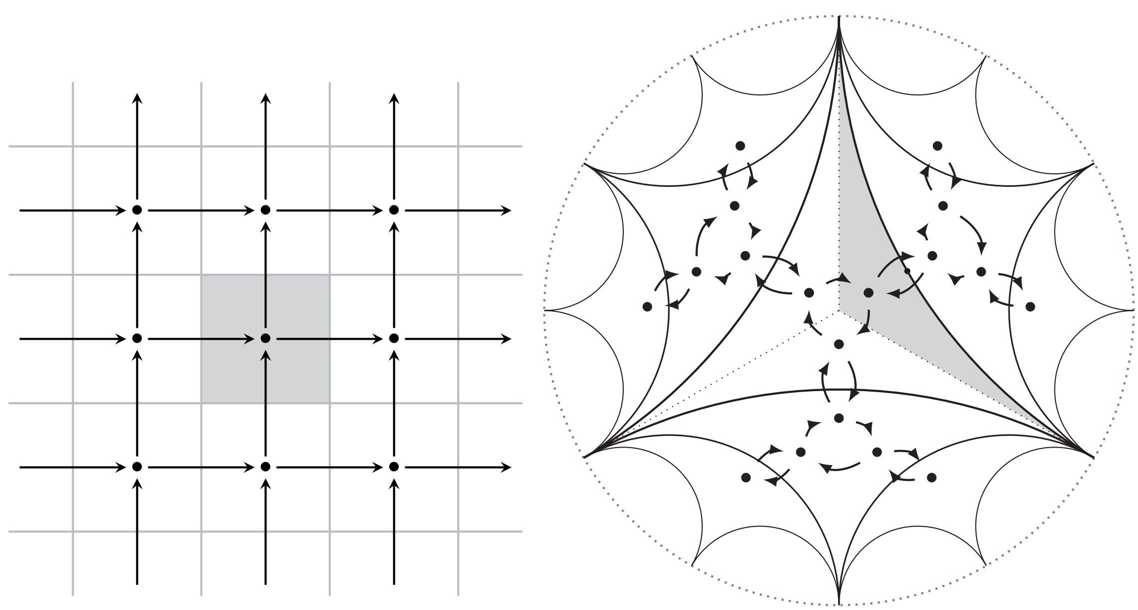

FIGURE 4. Skeleton illustrations (for exchange graphs in moduli/stability spaces)

4. Further direction

We (Akishi Ikeda and me) have started a new project 'q-stability conditions on Calabi-Yau $\mathbb{X}$ categories' ([IQ1, IQ2]), where we did the following:

- In [IQ1], we set up the framework of q-deformation of stability conditions on (Calabi-Yau-)$\mathbb{X}$ categories $\mathcal{D}_{\mathbb{X}}$, whose Grotendieck group is $R^{\oplus n}$ for $R = \mathbb{Z}[q^{\pm 1}]$.
- In [IQ2], we study the surface case and introduce the q-deformation of quadratic differentials to realize the q-deformation of stability conditions in this case.

In the forthcoming paper (together with Yu Zhou), we will study the decorated marked surfaces for Calabi-Yau-$\mathbb{X}$ categories.

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